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# GENERAL THEORY OF THE FACTORIZATION OF ORDINARY LINEAR DIFFERENTIAL OPERATORS

BY

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ABSTRACT. The problem of factoring the general ordinary linear differential operator $Ly = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_0 y$ into products of lower order factors is studied. The factors are characterized completely in terms of solutions of the equation $Ly = 0$ and its adjoint equation $L^*y = 0$. The special case when $L$ is formally selfadjoint of order $n = 2k$ and the factors are of order $k$ and adjoint to each other reduces to a well-known result of Rellich and Heinz: $L = Q^*Q$ if and only if there exist solutions $y_1, \cdots, y_k$ of $Ly = 0$ satisfying $W(y_1, \cdots, y_k) \neq 0$ and $[y_i; y_j] = 0$ for $i, j = 1, \cdots, k$; where $[\ ;\ ]$ is the Lagrange bilinear form of $L$.

**Introduction.** In this paper we investigate the problem of determining when the classical $n$th order linear differential operator

$$Ly = y^{(n)} + p_{n-1}y^{(n-1)} + \cdots + p_1 y' + p_0 y \tag{1}$$

can be factored into products of lower order operators of the same type and how these factors may be characterized.

In §1 we collect results from the elementary theory of ordinary linear differential equations which we use in the subsequent development. §§2 and 3 deal with the cases of two or more than two factors respectively. Sufficient conditions on the coefficients for various types of factorizations are developed in §4. Some applications and illustrations are given in §5.

According to a well-known result of Pólya [12] the operator $L$ has a factorization into "products" of first order factors

$$Ly = r_n(r_{n-1} \cdots (r_1(r_0 y)')' \cdots)' \tag{2}$$

on some interval $I$ if and only if the equation $Ly = 0$ has a fundamental set of solutions $y_1, \cdots, y_n$ such that

$$W_k \neq 0 \quad \text{on } I \text{ for } k = 1, \cdots, n-1 \tag{3}$$

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where $W_1 = y_1$ and $W_k = \det[y_j^{(i-1)}]$ for $k = 2, \cdots, n-1$, $i, j = 1, \cdots, k$. For a short and elegant proof of this result see [14] or [3, pp. 91–92]. Such a factorization is closely related to disconjugacy and has received a lot of attention in the literature. For some recent papers see [6], [9], [15], [19].

In [16] it is shown that $L$ has a factorization of type

$$L = PQ \tag{4}$$

on some interval $I$ where $P$ and $Q$ are of the same type as $L$ of orders $n-k$ and $k$ respectively if and only if there exist $k$ linearly independent solutions $y_1, \cdots, y_k$ of $Ly = 0$ whose Wronskian $W_k = W(y_1, \cdots, y_k)$ satisfies

$$W_k = W(y_1, \cdots, y_k) \neq 0 \quad \text{on } I. \tag{5}$$

In general not much seems to be known about factorizations of type (4). A notable exception is the work of Rellich and Heinz given in [7] and the paper of Kreĭn [8]. See also [4], [3] and [19].

For a result on multiple second order factors see Miller [11].

Explicit conditions on the coefficients which yield factorizations of types (2) and (4) were obtained in [19] for $n = 3$ and, for type (4) only, in [17] for general $n$.

1. Throughout the paper we will assume that the coefficients $p_i$ in (1) are continuous complex valued functions defined on some interval $I$. Smoothness conditions will be assumed as needed. Unless explicitly stated otherwise the interval $I$ can be any nondegenerate subinterval of the real line: open, closed, half-open, finite or infinite.

We proceed to list some facts from the theory of ordinary linear differential equations. For proofs the reader is referred to any of the standard books on the subject. We mention specifically the books by Coddington and Levinson [2], Miller [11] and Hartman [5].

Denote by $N(L)$ the set of all solutions of $Ly = 0$.

**Lemma 1.** *The set $N(L)$ is an $n$-dimensional vector space. If $\{y_1, \cdots, y_n\}$ is a basis of $N(L)$, then $W(y_1, \cdots, y_n) \neq 0$ and*

$$Ly = W(y_1, \cdots, y_n, y)/W(y_1, \cdots, y_n) \quad \textit{for all } y \in C^n. \tag{6}$$

If we assume that $p_i \in C^i$ for $i = 0, \cdots, n-1$ then the operator

$$L^*y = (-1)^n y^{(n)} + (-1)^{n-1}(\bar{p}_n y)^{(n-1)} + \cdots + (-1)(\bar{p}_1 y)' + \bar{p}_0 y$$

can be put into the same form as $L$. The operator $L^*$ is called the formal adjoint of $L$ and $L$ is said to be formally selfadjoint if $L^* = L$. (By using appropriate quasidifferential expressions one can avoid making any differentiability assumptions on the coefficients and still develop the adjoint operator—see [18]. However we will not do that here.)

This leads us to the Lagrange identity which will play an important role later.

**Lemma 2.** *For any $u$, $v$ in $C^n$ we have*

$$\bar{v} Lu - \overline{uL^* v} = [u; v]_L$$

*where* $[u; v]_L = \sum_{i=0}^{n} \sum_{j=0}^{i-1} (-1)^j (p_i \bar{v})^{(j)} u^{(i-1-j)}$ *with* $p_n \equiv 1$.

The form $[u; v]_L$ is called the concomitant of $L$ and is bilinear: For any constants $c_i$ we have

$$\left[\sum_{i=1}^{k} c_i u_i; v\right]_L = \sum_{i=1}^{k} c_i [u_i; v]_L \quad \text{and} \quad \left[u; \sum_{i=1}^{k} c_i v_i\right]_L = \sum_{i=1}^{k} \bar{c}_i [u; v_i]_L.$$

**Corollary.** *If $u \in N(L)$ and $v \in N(L^*)$, then $[u; v]_L$ is constant. Also $[u; v]_L = 0$ if and only if $[v; u]_{L^*} = 0$.*

**Definition.** Given $u \in N(L)$ and $v \in N(L^*)$ we say that $v$ is conjugate to $u$ if $[u; v]_L = 0$. Note that $v$ is conjugate to $u$ if and only if $u$ is conjugate to $v$. Given $y_1, \cdots, y_k \in N(L)$ and $v$ in $N(L^*)$ we say that $v$ is conjugate to $\{y_1, \cdots, y_k\}$ if $v$ is conjugate to $y_i$ for each $i = 1, \cdots, k$.

**Remark.** From the Corollary to Lemma 2 it follows that if each of $v_1, \cdots, v_r$ is conjugate to $\{y_1, \cdots, y_k\}$ then every member of the subspace generated by $v_1, \cdots, v_r$ is conjugate to every member of the subspace generated by $y_1, \cdots, y_k$. So the property of being conjugate is a property that subspaces of $N(L^*)$ may have relative to subspaces of $N(L)$—not just particular elements. A particularly simple way to construct conjugate elements is as follows: If $u \in N(L)$ and $v \in N(L^*)$ satisfy the initial conditions $u^{(i)}(a) = 0$ for $i = 0, \cdots, k$ and $v^{(j)}(a) = 0$ for $j = 0, \cdots, n-k$; then $[u; v]_L = [u; v]_L(a) = 0$.

For each $s \in I$, let $V(\cdot, s)$ be the solution of $Ly = 0$ which satisfies the initial conditions $y^{(i)}(s) = \delta_{i,n-1}$ for $i = 0, \cdots, n-1$. This function $V(t, s)$ is called the Cauchy function of $L$. Some of its basic properties are listed in

**Lemma 3.** (a) *For any $f \in C$ the solution $y$ of $Ly = f$ which satisfies the initial conditions $y^{(i)}(a) = 0$ for $i = 0, \cdots, n-1$ is given by the formula*

$$y(t) = \int_a^t V(t, s) f(s) ds \quad \text{for } t \in I. \tag{7}$$

(b) *Assume* $p_i \in C^i$ *so that* $L^*$ *can be formed. Given a basis* $y_1, \cdots, y_n$ *of* $N(L)$ *there exists a basis* $y_1^*, \cdots, y_n^*$ *of* $N(L^*)$ *such that*

$$V(t, s) = y_1(t)\bar{y}_1^*(s) + \cdots + y_n(t)\bar{y}_n^*(s) \quad \text{for all } t, s \in I.$$

(c) $(-1)^{n-1}\bar{V}(s, t)$ *is the Cauchy function of* $L^*$.

The next lemma shows how to obtain a basis of $N(L^*)$ from a given basis of $N(L)$.

**Lemma 4.** *Suppose* $y_1, \cdots, y_n$ *form a basis of* $N(L)$.
*Let*

$$\bar{z}_i = W(y_1, \cdots, \hat{y}_i, \cdots, y_n)/W(y_1, \cdots, y_n) \tag{8}$$

*for* $i = 1, \cdots, n$ *where the circumflex over* $y_i$ *indicates that* $y_i$ *is missing.*
*Then* $z_1, \cdots, z_n$ *form a basis of* $N(L^*)$.

2. **The factorization** $L = RQ$. We consider the problem of determining when the operator $L$ can be factored i.e. can be represented by: $L = RQ$ where $R$ and $Q$ are operators of lower order. First we make precise the meaning of such a representation.

Given an operator $Q$ of the form $Qy = y^k + q_{k-1}y^{(k-1)} + \cdots + q_0 y$ with $q_i \in C$, certainly $Qy$ will make sense and be a continuous function, for any $y \in C^k$. If $1 \le k < n$, $q_i \in C^{n-k}$ for $i = 0, 1, \cdots, k-1$ and $R$ is an operator of type $Ry = y^{(n-k)} + r_{n-k-1}y^{(n-k-1)} + \cdots + r_0 y$ with $r_i \in C$ then $RQ$ can be defined by $(RQ)(y) = R(Qy)$ for every $y \in C^n$.

So by $L = RQ$ we simply mean that $Ly = R(Qy)$ for every $y \in C^n$.

By a direct computation $RQ$ can be put in the form (1) i.e. $RQy = y^{(n)} + s_{n-1}y^{(n-1)} + \cdots + s_0 y$ and it follows immediately that $L = RQ$ if and only if $p_i = s_i$ for $i = 0, \cdots, n-1$. We list only the first couple of these equations:

$$p_{n-1} = q_{k-1} + r_{n-k-1} = s_{n-1},$$

$$p_{n-2} = q'_{k-1} + q_{k-2} + r_{n-k-1}q_{k-1} + r_{n-k-2} = s_{n-2}, \text{ etc.}$$

By solving these equations successively it is apparent that given the $p$'s and $q$'s (with the $q$'s satisfying $q_i \in C^{n-k}$, $i = 0, 1, \cdots, k-1$), the $r$'s are determined uniquely; and given the $p$'s and $r$'s, the $q$'s are determined uniquely. In other words, given a factorization $L = RQ$, $Q$ determines $R$ uniquely and $R$ determines $Q$ uniquely.

Moreover the differentiability properties of the coefficients can be readily read off from these equations as well. For instance if $p_i \in C^i$ for $i = 0, \cdots, n-1$; then $r_{n-k-j} \in C^{n-j} \cap C^{n-k}$ for $j = 1, \cdots, n-k$. So $L^*$, $R^*$ and $Q^*R^*$ are all defined. Consequently $L = RQ$ implies $L^* = Q^*R^*$—see [11].

We now state the result from [16].

**Theorem 1.** *Suppose $1 \leq k < n$. Then the factorization $L = RQ$ where $Q$ is an operator of the form $Qy = y^k + q_{k-1}y^{k-1} + \cdots + q_0 y$, with $q_i \in C^{n-k}$, holds if and only if there exist $k$ solutions $y_1, \cdots, y_k$ of $Ly = 0$ which satisfy* (5) *i.e. $W(y_1, \cdots, y_k) \neq 0$. Furthermore $Q$ has the representation* (6).

The factorization $L = RQ$ holds on any interval where the Wronskian condition $W(y_1, \cdots, y_k) \neq 0$ is satisfied and conversely. Note that (5) is always satisfied locally: determine a fundamental set of solutions $y_1, \cdots, y_n$ by the initial conditions $y_j^{i-1}(a) = \delta_{ij}$ for $i, j = 1, \cdots, n$. Then $W(y_1, \cdots, y_k)(a) = 1$ and therefore $W(y_1, \cdots, y_k)$ is positive in a neighborhood of $a$, since it is a continuous function.

Since the proof of Theorem 1 is short we include it here for the sake of completeness. Suppose $L = RQ$. Since any solution of $Qy = 0$ is also a solution of $Ly = 0$ we need only choose $y_1, \cdots, y_k$ to be a fundamental set of solutions of $Qy = 0$ to get $W(y_1, \cdots, y_k) \neq 0$. The differentiability conditions of the coefficients $q_i$ can be read off the representation of $Q$ given by Lemma 1.

On the other hand suppose $W(y_1, \cdots, y_k) \neq 0$. Define

$$Qy = W(y_1, \cdots, y_k, y)/W(y_1, \cdots, y_k) \quad \text{for } y \in C^n.$$

Then $q_i \in C^{n-k}$ for $i = 0, \cdots, k-1$. Letting $Ry = y^{(n-k-1)} + \cdots + r_0 y$, computing $R(Qy)$ and setting the coefficients of $y^{(n-1)}, y^{(n-2)}, \cdots, y^{(k)}$ equal to $p_{n-1}, p_{n-2}, \cdots, p_k$, respectively, yields $n-k$ equations (the first two of which are listed above) which can be solved successively starting with the equation from the coefficient of $y^{(n-1)}$. Setting $N = L - RQ$ we show that $N = 0$. The order of $N$ is less than $k$ since the coefficients of $y^{(j)}$ for $j \geq k$ are all zero. But $Ny_i = Ly_i - R(Qy_i) = 0$ for $i = 1, \cdots, k$, hence $N = 0$ and so $L = RQ$.

For the remainder of the paper and mainly as a matter of convenience we assume that $p_i \in C^i$ for $i = 0, \cdots, n-1$.

As stated in the remark preceding Theorem 1, $L = RQ$ implies $L^* = Q^*R^*$ and conversely, since $M^{**} = M$, we have that $L^* = Q^*R^*$ implies $L = RQ$. From this observation and Theorem 1 we conclude that there exist $y_1, \cdots, y_k \in N(L)$ satisfying $W(y_1, \cdots, y_k) \neq 0$ if and only if there exist $z_1, \cdots, z_{n-k}$ in $N(L^*)$ satisfying $W(z_1, \cdots, z_{n-k}) \neq 0$. We now investigate this relationship between solutions of an equation and its adjoint. This relationship is actually between subspaces of $N(L)$ of dimension $k$ and subspaces of $N(L^*)$ of dimension $n-k$.

How are $z_1, \cdots, z_{n-k}$ determined in terms of $y_1, \cdots, y_k$?

To answer this question we need a lemma which may be of independent interest.

Suppose $y_1, \cdots, y_k \in N(L)$ and satisfy (5). Since $y_1, \cdots, y_k$ are linearly independent, there exist elements $y_{k+1}, \cdots, y_n$ from $N(L)$ such that $y_1, \cdots, y_k$, $y_{k+1}, \cdots, y_n$ is a basis of $N(L)$. Let $z_i$ be defined by (8) for $i = 1, \cdots, n$.

**Lemma 5.** *The Lagrange bracket* $[y_j; z_i]_L$ *is zero for* $i \neq j$ *and nonzero for* $i = j$, $i, j = 1, \cdots, n$.

**Proof.** From the expansion of the Wronskian determinant $W = W(y_1, \cdots, y_n)$ along columns we obtain:

$$\sum_{q=1}^{n} (-1)^{q+1} M(y_i^{(q-1)}) y_j^{(q-1)}/W = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{9}$$

where $M(y_i^{(q-1)})$ is the minor of the element $y_i^{(q-1)}$. Using (9) and the formula for the Lagrange bracket given in Lemma 2 (with $u = y_j$ and $v = z_i$) and setting coefficients of $y_m^{(q)}$ equal we need to show:

$$\begin{aligned} (-1)^{n+r} M(y_j^{(n-r)})/W &= p_{n-r+1} z_i - (p_{n-r+2} z_i)' + (p_{n-r+3} z_i)'' \\ &\quad + \cdots + (-1)^{r-2}(p_{n-1} z_i)^{(r-2)} + (-1)^{r-1} z_i^{(r-1)} \end{aligned} \tag{10}$$

$$\text{for } r = 1, 2, \cdots, n.$$

The case $r = 1$ with $p_n$ taken to be the constant 1 is evident from the definition of $z_i$.

Assume we have established (10) for a particular $r$, $1 \leq r < n$. Then solving for $z_i^{(r-1)}$ and differentiating yields:

$$\begin{aligned} (-1)^{r-1} z_i^{(r)} &= (-1)^{n+r}(M(y_j^{(n-r)})/W)' - (p_{n-r+1} z_i)' \\ &\quad + (p_{n-r+2} z_i)'' - (p_{n-r+3} z_i)''' + \cdots + (-1)^{r-1}(p_{n-1} z_i)^{(r-1)}. \end{aligned}$$

Using Abel's formula $W' = -p_{n-1} W$ and the formula for the derivative of a determinant we get

$$\begin{aligned} (M(y_j^{(n-r)})/W)' &= W^{-2}[(M(y_j^{(n-r)}))' - M(y_j^{(n-r)}) W'] \\ &= W^{-1}[(M(y_j^{(n-r)}))' + p_{n-1} M(y^{(n-r)})] \\ &= W^{-1}[M(y_j^{(n-r-1)}) - p_{n-1} M(y^{(n-r)}) \\ &\qquad + (-1)^{r+1} p_{n-r} W(y_1 \cdots \hat{y}_i \cdots y_n) + p_{n-1} M(y_j^{(n-r)})] \\ &= M(y_j^{(n-r-1)})/W + (-1)^{r+1} p_{n-r} z_i. \end{aligned}$$

Substituting into the above expression for $z_i^{(r)}$ yields:

$$(-1)^{r-1}z_i^{(r)} = (-1)^{n+r}[M(y_j^{(n-r-1)})/W + (-1)^{r-1}p_{n-r}z_i]$$
$$- (p_{n-r+1}z_i)' + (p_{n-r+2}z_i)'' - \cdots + (-1)^{r-1}(p_{n-1}z_i)^{(r-1)}.$$

Finally, solving for $M(y_j^{(n-r-1)})/W$ we obtain:

$$(-1)^{n+r}W^{-1}M(y_j^{(n-r-1)}) = (-1)^{n+r+1}(-1)^{r+1}p_{n-r}z_i + (p_{n-r+1}z_i)' - (p_{n-r+2}z_i)''$$
$$+ \cdots + (-1)^r(p_{n-1}z_i)^{(r-1)} + (-1)^{r-1}z_i^{(r)}.$$

This is formula (10) with $r$ replaced by $r+1$ and our proof of Lemma 5 is complete.

It follows directly from Lemma 5 that $z_j$ is conjugate to $\{y_1, \cdots, y_k\}$ for each $j = k+1, \cdots, n$. Consequently, by the remark following Lemma 2, the subspace of $N(L^*)$ generated by $z_{k+1}, \cdots, z_n$ is conjugate to the subspace of $N(L)$ generated by $y_1, \cdots, y_k$. It will be shown later that no other elements of $N(L^*)$ are conjugate to $\{y_1, \cdots, y_k\}$.

**Lemma 6.** *Suppose $L = RQ$ and $y_1, \cdots, y_k$ is a basis of $N(Q)$. Let $V(x, t)$ be the Cauchy function of $Q$ and choose a basis $\bar{y}_1^*, \cdots, \bar{y}_k^*$ of $N(Q^*)$ such that $V(x, t) = y_1(x)y_1^*(t) + \cdots + y_k(x)y_k^*(t)$. Then*

$$R^*z = \sum_{i=1}^{k} (-1)^n\overline{[y_i; z]}_L\bar{y}_i^* \quad \textit{for every } z \in C^n. \tag{11}$$

**Proof.** Let $a \in I$, fix $t \in I$ and let $u(x) = V(x, t)$ for all $x$ in $I$. Then, taking $v = z$ in the Lagrange identity of Lemma 2 and integrating from $a$ to $t$ we obtain

$$-\int_a^t V(x, t)\overline{(L^*z)}(x)dx = [V(\cdot\,; t); z]_L\Big|_{t=a}^{t}$$
$$= \sum_{i=1}^{k} y_i^*(t)[y_i; z]_L(t) - \sum_{i=1}^{k} y_i^*(t)[y_i; z]_L(a)$$

for each $t \in I$.

Since $(-1)^{n+1}\bar{V}(t, x)$ is the Cauchy function of $Q^*$ we have:

$$Q^*\left\{\int_a^t (-1)^{n+1}\bar{V}(x, t)f(x)dx\right\} = f(t) \quad \text{for all } t \in I \text{ and } f \in C.$$

Taking $f = L^*z$ and using $\bar{y}_i^* \in N(Q^*)$ for $i = 1, \cdots, k$ yields $L^*z = Q^*\{\Sigma_{i=1}^k(-1)^n\bar{y}_i^*\overline{[y_i; z]}_L\}$. Letting $Mz = \Sigma_{i=1}^k(-1)^n\bar{y}_i^*\overline{[y_i; z]}_L$ we have shown that $L^*z = Q^*Mz$ for every $z \in C^n$. Since $L^* = Q^*R^*$ and $R^*$ is unique we conclude that $M = R^*$ and the proof is complete.

**Theorem 2.** *Let $y_1, \cdots, y_n$ be a basis of $N(L)$ with $y_1, \cdots, y_k$ for $1 \leq k < n$ satisfying* (5). *Let $z_i$ be defined by* (8) *for $i = 1, \cdots, n$. Suppose $L = RQ$ with $Q$ given by* (6) *with $n = k$ and $L = Q$. Then $z_{k+1}, \cdots, z_n$ form a basis of $N(R^*)$ and*

$$R^*z = W(z_{k+1}, \cdots, z_n, z)/W(z_{k+1}, \cdots, z_n) \quad \text{for } z \in C^n. \tag{12}$$

**Proof.** The representation (12) follows from Lemma 1, once it is established that $z_j$ for $j = k+1, \cdots, n$ are in $N(R^*)$ and are linearly independent. The linear independence follows from the fact that $z_1, \cdots, z_n$ are a basis of $N(L^*)$.

By Lemma 5 we have $[y_i;\ z_j]_L = 0$ for $i = 1, \cdots, k$ and $j = k+1, \cdots, n$. Hence, by Lemma 6,

$$R^*z_j = \sum_{i=1}^{k} (-1)^n \bar{y}_i^* \overline{[y_i, z_j]_L} = 0 \quad \text{for } j = k+1, \cdots, n.$$

This completes the proof.

**Corollary.** *Suppose $n = 2k$ and the hypothesis and notation of Theorem 2 hold. Then $R^* = Q$ if and only if*

$$z_j \in \operatorname{span}\{y_1, \cdots, y_k\} \quad \text{for } j = k+1, \cdots, n.$$

The special case of this Corollary when $L$ is formally selfadjoint is a result due to Rellich and Heinz—see Heinz [7, Satz 3 and Zusatz p. 16]. See also Coppel [3, Theorem 19, p. 80] and Kreĭn [8].

We conclude this section by showing that Lemma 5 characterizes all the elements of $N(L^*)$ which are conjugate to $\{y_1, \cdots, y_k\}$, thus establishing the sentence just above Lemma 6.

**Remark.** Suppose $y_1, \cdots, y_k$ are in $N(L)$ satisfying (5) and $z_i$ for $i = 1, \cdots, n$ are defined by (8). Let $z \in N(L^*)$. Then $[y_i;\ z]_L = 0$ for all $i = 1, \cdots, k$ if and only if $z$ is a linear combination of $z_{k+1}, \cdots, z_n$.

**Proof.** The if part has already been established. Suppose $[y_i;\ z]_L = 0$ for $i = 1, \cdots, k$. By Lemma 6, $R^*z = 0$ and hence $z$ is a linear combination of $z_{k+1}, \cdots, z_n$ since these form a basis for $N(R^*)$.

We conclude this section with another

**Remark.** If we are dealing with a differential operator $My = r_n y^{(n)} + \cdots + r_0 y$ with $r_n(t) \neq 0$ on $I$, then $L = r_n^{-1}M$ has the form (1). We can then apply our factorization results to $L$.

If $(1/r_n)M = RQ$, then $M = r_n RQ$ where $R$ and $Q$ have leading coefficient 1. We observe that $r_n$ can be "absorbed" by either $R$ or $Q$ or both to yield a factorization $M = R'Q'$ where the leading coefficients of $R'$, $Q'$, say $r'$, $q'$ have the property $r'q' = r_n$.

3. **Multiple factors.** In this section we investigate the factorization of $L$ into products of more than two factors. For this purpose we introduce two weak forms of the well-known property W introduced by Pólya in [12]: Properties EW and OW. We say that a linear differential operator $L$—or equivalently its null space $N(L)$—has property EW [OW] if there exists a basis $y_1, y_2, \cdots, y_n$ of $N(L)$ with the property that all even [odd] order Wronskians have no zero on $I$ i.e. $W(y_1, \cdots, y_k)(t) \neq 0$ for all $t$ in $I$ and for $k$ even [odd].

A basis $y_1, \cdots, y_n$ of $N(L)$ is called an EW [OW] system of $L$ if $W(y_1, \cdots, y_k) \neq 0$ for all even [odd] values of $k < n$.

Only $k < n$ is significant since the Wronskian of a basis is always nonzero. We remark that an operator $L$ can have both properties EW and OW without having property W i.e. without (3) holding. Of course, in this case EW and OW hold with respect to different bases. A simple example is given.

**Example.** Let $Ly = y''' - y'' + y' - y$. A basis of $N(L)$ is: $y_1(x) = e^x$, $y_2(x) = \sin x$, $y_3(x) = \cos x$. Clearly $W(y_1) = e^x \neq 0$ and $W(y_2, y_3)(x) = -1 \neq 0$ so that $L$ has property OW and EW. But $L$ cannot have property W since it is not disconjugate on, say $I = [0, \infty)$, $\sin x$ being a nontrivial solution with infinitely many zeros.

**Theorem 3.** *Suppose $y_1, \cdots, y_n$ is a basis of $N(L)$. Let $z_i$ be defined by (8) for $i = 1, \cdots, n$.* (i) *If $n$ is even, then $y_1, \cdots, y_n$ is an* EW [OW] *system of $L$ if and only if $z_1, \cdots, z_n$ is an* EW [OW] *system of $L^*$.* (ii) *If $n$ is odd, then $y_1, \cdots, y_n$ is an* EW [OW] *system of $L$ if and only if $z_1, \cdots, z_n$ is an* OW [EW] *system of $L^*$.*

**Proof.** In §2 we saw that $W(y_1, \cdots, y_k) \neq 0$ implies that $L$ has a factorization $L = RQ$ with $y_1, \cdots, y_k$ being a basis of $N(Q)$ and $z_{k+1}, \cdots, z_n$ being a basis of $N(R^*)$. Hence $W(z_{k+1}, \cdots, z_n) \neq 0$ and so half the theorem follows since $n - k$ is even for $n$ and $k$ even and odd for $n$ odd and $k$ even. The other half follows by interchanging the roles of $L$ and $L^*$.

**Theorem 4.** (i) *Suppose $n$ is even i.e. $n = 2k$. If $L$ has property* EW, *then $L$ is a product of $k$ second order operators each of type: $y'' + r_1y' + r_0y$; i.e. $L = Q_1Q_2 \cdots Q_k$ where each $Q_i$ is a second order operator with leading coefficient* 1. *If $L$ has property* OW, *then $L$ has a factorization $L = PQ_1Q_2 \cdots Q_{k-1}R$ where each $Q_i$ is a second order operator and $P$ and $R$ are first order operators and all have leading coefficient* 1.

(ii) *Suppose $n$ is odd, say $n = 2k + 1$. If $L$ has property* EW *then $L$ has a* *factorization* $L = PQ_1 \cdots Q_k$ *where each* $Q_i$ *is of order* 2 *and* $P$ *has order* 1 *and all have leading coefficient* 1. *If* $L$ *has property* OW *then* $L$ *has a factorization* $L = Q_1Q_2 \cdots Q_kR$ *where each* $Q_i$ *has order* 2, $R$ *has order* 1 *and all have leading coefficient* 1.

**Proof.** We prove the first part of (i) only since the proofs of the other parts are similar. Let $y_1, \cdots, y_n$ be an EW system of $L$. Then by Theorem 1, $L = Q_1Q$ where $Q_1$ is of order 2 with leading coefficient 1, $Q$ is of order $n-2$ with leading coefficient 1 and has the representation (6) given in Lemma 1 for $k = n-2$ i.e.

$$Q(y) = W(y_1, \cdots, y_{n-2}, y)/W(y_1, \cdots, y_{n-2}) \quad \text{for all } y \in C^n.$$

Since $y_1, \cdots, y_{n-2}$ are in $N(Q)$ and $W(y_1, \cdots, y_{n-4}) \neq 0$ we can apply Theorem 1 to $Q$ and obtain the factorization $Q = Q_2R$ where $Q_2$, $R$ have leading coefficient 1, $Q_2$ is of order 2 and $R$ of order $n-4$. This yields: $L = Q_1Q_2R$. Repeating the above procedure we obtain the desired factorization of $L$: $L = Q_1Q_2 \cdots Q_k$.

All of the second order factors $Q_i$ appearing in Theorem 4 can be made formally selfadjoint at the expense of introducing a positive function as multiplier. This result is stated more precisely as

**Corollary.** (i) *Suppose* $n$ *is even i.e.* $n = 2k$. *If* $L$ *has property* EW, *then* $L = rP_1 \cdots P_k$ *where* $r$ *is a positive function and each* $P_i$ *is a second order formally selfadjoint operator i.e. each* $P_i$ *has order* 2 *and* $P_i^* = P_i$ *for each* $i = 1, \cdots, k$. *If* $L$ *has property* OW, *then* $L = rSP_1 \cdots P_kR$ *where* $r$ *is a positive function,* $S$, $R$ *are first order operators and each* $P_i$ *is a second order formally selfadjoint operator.* (ii) *Suppose* $n$ *is odd*: $n = 2k+1$. *Then* $L$ *having property* EW *implies* $L = rSP_1 \cdots P_k$ *and* OW *implies* $L = rP_1 \cdots P_kR$ *where* $r_1S$, $R$, $P_i$ *are as in* (i).

**Proof.** Again we prove only the first part of (i) since the other proofs are similar. Let $L = Q_1 \cdots Q_k$ be the factorization of Theorem 4. Let $Q_ky = y'' + \alpha y' + \beta y$. Taking $P_k = r_kQ_k$ with $r_k = \exp[\int \alpha]$ we get $P_k^* = P_k$. Let $Q_{k-1}y = y'' + ay' + by$, then

$$Q_{k-1}r_k^{-1}r_kQ_k(y) = Q_{k-1}r_k^{-1}P_k(y) = r_k^{-1}(P_ky)'' + [ar_k^{-1} + 2(r_k^{-1})'](P_ky)'$$
$$+ [(r_k^{-1})'' + a(r_k^{-1})' + br_k^{-1}]P_ky.$$

Letting $Sy = r_k^{-1}y'' + [ar_k^{-1} + 2(r_k^{-1})']y' + [(r_k^{-1})'' + a(r_k^{-1})' + br_k^{-1}]y$ we have $Q_{k-1}Q_ky = SP_ky$. Now writing

$$Sy = r_k^{-1}[y'' + r_k(ar_k^{-1} + 2(r_k^{-1})')y' + r_k((r_k^{-1})'' + a(r_k^{-1})' + br_k^{-1})y]$$

and letting $r_{k-1} = \exp[\int(a + 2r_k(r_k^{-1})')]$, we get $Q_{k-1}Q_k y = r_k^{-1}r_{k-1}^{-1}P_{k-1}P_k y$ where $P_{k-1}y = r_{k-1}[y'' + (a + 2r_k(r_k^{-1})')y' + r_k^{-1}((r_k^{-1})'' + a(r_k^{-1})' + br_k^{-1})y]$ and $P_{k-1}^* = P_{k-1}$. Continuing in this way we end up with $L = r_k^{-1}r_{k-1}^{-1}\cdots r_1^{-1}P_1\cdots P_k$ concluding the proof.

4. In this section we give sufficient conditions on the coefficients $p_i$ for various kinds of factorizations to hold.

**Property** $\mathrm{W}_k$. We say that $L$ has property $\mathrm{W}_k$ if there exist $k$ solutions of $Ly = 0$ satisfying $W(y_1, \cdots, y_k) \neq 0$.

We start by recalling some known results:

(1) If $p_i \leq 0$ for $i = 0, 1, \cdots, n-2$; then $L$ has property $\mathrm{W}_1$.

(2) If $(-1)^{n-i}p_i \leq 0$ for $i = 0, 1, \cdots, n-2$; then $L$ has property $\mathrm{W}_1$. Note that no sign condition is needed on $p_{n-1}$. Also we remark that $L$ having property $\mathrm{W}_1$ is equivalent to $L^*$ having property $\mathrm{W}_{n-1}$.

(3) Suppose $p_i \equiv 0$ for $i = 1, 2, \cdots, n-3$ and $p_{n-2} \leq 0$. Then $L$ has property EW if $p_0 \geq 0$ and $L$ has property OW if $p_0 \leq 0$.

(4) If $(-1)^{n+1-j}p_j \geq 0$ for $j = 0, 1, \cdots, n-2$; then $L$ has property $\mathrm{W}_{n-1}$ (and hence $\mathrm{W}_1$ holds for $L^*$).

Results 1 and 2 are well known—for proofs see [17] for 1 and [5, p. 508] for 2. For 3 and 4 see [17].

Consider $My = (py^{(n)})^{(n)} + (ry^{(n-1)})^{(n-1)} + qy$ with $p > 0$ and $p \in C^n$, $r \in C^{n-1}$ and $q \in C$.

**Theorem 5.** *Suppose $r \leq 0$ (in addition to $p > 0$). Determine solutions $y_1, \cdots, y_{2n}$ by the initial conditions $y_j^{(i-1)}(a) = \delta_{ij}$ for $i, j = 1, \cdots, 2n$. Then $y_1, \cdots, y_{2n}$ is an* EW *system if $q \geq 0$ and an* OW *system if $q \leq 0$ on any interval $(a, b)$ with $b > a$.*

**Proof.** Since the proof is similar to that given in [17] for result 3 above we merely outline it here. Since $M$ is formally selfadjoint we can restrict ourselves to $k \leq n$. Just as in [17] we construct a vector matrix system $Y' = FY$ where $Y$ is a vector of order $\binom{2n}{k}$ whose components are $k$th order Wronskians of $y_1, \cdots, y_k$ and their quasi derivatives up to the $(2n-1)$th derivative. These determinants are ordered lexicographically starting with $W(y_1, \cdots, y_k)$. The $\binom{2n}{k}$ by $\binom{2n}{k}$ matrix $F$ has entries from the set $\{0, 1, 1/p, -r, (-1)^k q\}$. Since all entries of the matrix $F$ are nonnegative and all components of $Y$ are nonnegative at the point $a$ with the first component $W(y_1, \cdots, y_k)(a) = 1$, it follows from a lemma of Mikusinski [10], that all entries of $Y$ are nonnegative to the right of the point $a$

and in particular $W(Y_1, \cdots, y_k)(t) > 0$ for $t > a$.

Recently, Ridenhour [13] has obtained sufficient conditions, expressed as $n$ inequalities involving the coefficients and their derivatives, for an operator of order $2n$ to have property $W_n$.

5. **Applications.** In this final section we mention some applications. Determine a fundamental set of solutions $y_1, \cdots, y_n$ of $Ly = 0$ by the initial conditions $y_j^{(i-1)}(a) = \delta_{ij}$ for $i, j = 1, \cdots, n$ and $a \in I$.

Suppose $y$ is a nontrivial solution of $Ly = 0$ which has a zero of order $k$ at $a$. Then $y$ has no zero of order $n-k$ to the right of $a$ if and only if $W(y_{k+1}, \cdots, y_n)(t) \neq 0$ for $t > a$.

**Proof.** Such a solution $y$ must be a nontrivial linear combination of $y_{k+1}, \cdots, y_n$. Hence $y$ has a zero of order $n-k$ at some point $b > a$ if and only if $W(y_{k+1}, \cdots, y_n)(b) = 0$.

Thus, if $y_n, \cdots, y_1$ is an EW system, a nontrivial solution can have a zero of order $k$ at $a$ and a zero of order $n-k$ at $b > a$ only if $n-k$ is odd. Similarly if $y_n, \cdots, y_1$ is an OW system a nontrivial solution can have a zero of order $k$ at $a$ and a zero of order $n-k$ at $b > a$ only if $n-k$ is even.

As a consequence of these observations we have: If $y_n, \cdots, y_1$ is either an EW system or an OW system then no nontrivial solution can have zeros at $a$, $b$ with $a < b$ of combined order greater than $n$ since this would imply that two Wronskians of consecutive integral order are zero at $b$ and clearly one of these must be even and one odd.

Combining the above remarks with explicit conditions on the coefficients given in §4 leads to the next two theorems.

**Theorem 6.** *Consider* $Ly = y^{(n)} + p_{n-1}y^{(n-1)} + p_{n-2}y^{(n-2)} + p_0 y$ *and assume* $p_{n-2} \leq 0$.

*If* $p_0 \geq 0$ $[p_0 \leq 0]$ *and* $y$ *is a nontrivial solution with zeros of order* $k$ *and* $n-k$ *at points* $a$, $b$ *with* $a < b$, *then* $n-k$ *is odd* [*even*]. *Consequently if* $p_0$ *does not change sign on* $I$ *then no nontrivial solution has zeros at* $a$, $b \in I$ *with* $a < b$ *of combined order* $> n$.

**Theorem 7.** *Consider* $My = (py^{(n)})^{(n)} + (ry^{(n-1)})^{(n-1)} + qy$ *with* $p > 0$, $p \in C^n$, $r \in C^{n-1}$, $q \in C$.

*If* $q \geq 0$ $[q \leq 0]$ *and* $y$ *is a nontrivial solution with zeros of order* $k$ *and* $n-k$ *at points* $a$, $b$ *with* $a < b$; *then* $n-k$ *is odd* [*even*].

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